SELF-SIMILAR SOLUTION OF THE NAVIER-STOKES EQUATIONS FOR A COMPLETELY IONIZED HYDROGEN PLASMA (THE PLANE PISTON PROBLEM)

R. G. Luk'yanova and S. I. Fadeev

The self-similar motion of a completely ionized hydrogen plasma is considered in the two-temperature hydrodynamic approximation, i.e., we consider the plane piston problem and the problem on energy release at a fixed wall. Results obtained by numerical integration of the relevant system of ordinary differential equations are quoted.

The undisturbed medium is assumed to be a dense, completely ionized hydrogen plasma at rest, with density ρ_N and zero electron and ion temperatures. We consider the plane piston problem for a given heat regime. The piston motion and the heat regime are chosen in such a way that self-similar motion of the medium results [1]. We neglect the role of radiation and assume a zero magnetic field.

1. The perturbations of a dense completely ionized plasma are commonly described by a system of Navier-Stokes equations, the general form of which is given in [2], as an example. Under the assumption of strict electrical neutrality, an ion charge z = 1, and a ratio of specific heats $\gamma = C_p/C_V = 5/3$, the system for uniform, plane, unestablished motion can be written as

$$\rho\frac{du}{dt}+\frac{\partial p_e}{\partial r}+\frac{\partial p_i}{\partial r}=\frac{4}{3}k_\mu\frac{\partial}{\partial r}\left[\left(\frac{p_i}{\rho}\right)^{5/2}\frac{\partial u}{dr}\right],\quad \frac{d\rho}{dt}+\rho\frac{\partial u}{\partial r}=0$$
$$\frac{3}{2}p_i\frac{d}{dt}\ln\frac{p_i}{\rho^{5/3}}=\frac{4}{3}k_\mu\left(\frac{p_i}{\rho}\right)^{5/2}\left(\frac{\partial u}{\partial r}\right)^2+k_q\rho^{5/2}\frac{p_e-p_i}{p_e^{3/2}} \tag{1.1}$$
$$\frac{3}{2}p_e\frac{d}{dt}\ln\frac{p_e}{\rho^{5/3}}=k_\varkappa\frac{\partial}{\partial r}\left[\left(\frac{p_e}{\rho}\right)^{5/2}\frac{\partial}{\partial r}\frac{p_e}{\rho}\right]+k_q\rho^{5/2}\frac{p_i-p_e}{p_e^{3/2}}$$

Here, t is time; r is a linear coordinate; u is the velocity of the ion gas, equal to the velocity of the electron gas; p_e and p_i are the pressures of the electron and ion gases; and k_μ, $k_\varkappa$, and k_q are constant factors, belonging, respectively, to the coefficient of ion viscosity, the coefficient of electron heat conduction, and the energy exchange between the electrons and ions. The dependences of k_μ, $k_\varkappa$, and k_q on the atomic constants are given by

$$k_\mu=\frac{0.72}{\sqrt{\pi}}\frac{m_i}{\lambda e^4},\quad k_\varkappa=\frac{2.37}{\sqrt{2\pi}}\frac{m_i^3}{\lambda e^4}\left(\frac{m_i}{m_e}\right)^{1/2},\ k_q=4\sqrt{2\pi}\frac{\lambda e^4}{m_i^3}\left(\frac{m_e}{m_i}\right)^{1/2} \tag{1.2}$$

Here, m_i is the ion mass; m_e is the electron mass; e is the elementary charge, and λ is the Coulomb logarithm. Both components of the medium, with ion temperature θ_i and electron temperature θ_e, are assumed to satisfy the equation of state of a perfect gas, i.e.,

$$p_i=\rho R\theta_i,\quad p_e=\rho R\theta_e,\quad R=k/m_i \tag{1.3}$$

where k is Boltzmann's constant.

Consider the case in which the piston motion and the heat energy release at the piston are specified by

$$r_*(t)=At^n,\qquad E(t)=Bt^\sigma \tag{1.4}$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 10, No. 3, pp. 34-39, May-June, 1969. Original article submitted November 20, 1968.

The constants defining the solution of the problem have the following dimensions:

$$[\rho_N] = ML^{-3}, \quad [A] = LT^{-n}, \quad [B] = MT^{-(2+\sigma)}$$
$$[k_\mu] = [k_\varkappa] = ML^{-6}T^4, \quad [k_q] = M^{-1}L^6T^{-4}$$

If we now require that only two of these constants have independent dimensionalities (say ρ_N and A), we find that

$$n = {}^4/_3, \qquad \sigma = 2 \tag{1.5}$$

In case (1.5), therefore, the motion of the medium is self-similar; and this latter may be used to transform (1.1) into a system of ordinary differential equations [1].

Notice that, in problems of plane (ν = 1), cylindrical (ν = 2), and spherical (ν = 3) pistons, with heat regimes of the (1.4) type, the motion is self-similar if

$$r_*(t) = At^{4/3}, \quad E(t) = Bt^{4/3(\nu+2)-2}$$

In particular, when A = 0 we obtain the problem concerning energy release at a point. In addition, the similarity property is easily seen to be retained with other values of z and γ.

2. As distinct from the motion of an ordinary viscous gas, in which viscosity and heat conduction are equally important factors (the Prandtl number is of order unity), in a high-temperature completely ionized plasma the Prandtl number becomes much smaller than unity because of the large electron heat conduction, and unless we are interested in the structure of the viscous jump, the influence of the ion viscosity can be neglected. Here, the continuous solution of system (1.1) transforms into a discontinuous solution with a so-called "isoelectron-thermal jump" [3]. It is assumed henceforth that the ion gas is nonviscous.

For convenience, after putting k_μ = 0 in (1.1), we rewrite the system in Lagrangian variables (x, t), where x is the mass coordinate:

$$\frac{\partial u}{\partial t} + \frac{\partial p_e}{\partial x} + \frac{\partial p_i}{\partial x} = 0, \qquad \frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial u}{\partial x} = 0$$
$$\frac{3}{2} R \frac{\partial \theta_i}{\partial t} = -p_i \frac{\partial u}{\partial x} + K\rho \frac{\theta_e - \theta_i}{\theta_e^{3/2}}, \qquad W = -\varkappa_0 \rho \theta_e^{5/2} \frac{\partial \theta_e}{\partial x} \tag{2.1}$$
$$\frac{3}{2} R \frac{\partial \theta_e}{\partial t} = -p_e \frac{\partial u}{\partial x} + K\rho \frac{\theta_i - \theta_e}{\theta_e^{3/2}} - \frac{\partial W}{\partial x}, \qquad \frac{p_i}{\theta_i} = \frac{p_e}{\theta_e} = \rho R$$

Here, W is the heat flux, determined by the electron heat conduction

$$\varkappa_0 = R^{7/2}k_\varkappa, \qquad K = 9.48R^3/\varkappa_0$$

Since the electron heat conduction is nonlinear, the solution of the problem takes the form, as in [4], of a temperature wave, traveling with finite velocity. Bearing this circumstance in mind, the initial and boundary conditions at t = 0, x > 0 and at the leading front of the disturbance with x = x_N, t > 0, are given by

$$\rho = \rho_N, \qquad u = \theta_i = \theta_e = W = 0 \tag{2.2}$$

and at the piston with x = 0, t > 0 by

$$u = {}^4/_3\, At^{1/3}, \quad W = 2Bt \quad \text{or} \quad u = {}^4/_3\, At^{1/3}, \quad \theta_e = T_0 t^{2/3} \tag{2.3}$$

In the notation of [4], the expressions for the transformation of system (2.1) may be written as

$$\begin{aligned} u(x,t) &= \alpha(S) R^{1/2} T_0^{1/2} t^{1/3}, & \theta_e(x,t) &= f_e(S) T_0 t^{2/3} \\ p_e(x,t) &= \beta_e(S) R^{-1} \varkappa_0 T_0^{5/2} t^{2/3}, & \theta_i(x,t) &= f_i(S) T_0 t^{2/3} \\ p_i(x,t) &= \beta_i(S) R^{-1} \varkappa_0 T_0^{5/2} t^{2/3}, & W(x,t) &= \varphi(S) R^{-3/2} \varkappa_0 T_0^3 t \\ \rho(x,t) &= \delta(S) R^{-2} \varkappa_0 T_0^{3/2}, & S &= xR^{3/2}\varkappa_0^{-1} T_0^{-2} t^{-4/3} \end{aligned} \tag{2.4}$$

After reduction of the transformed system to the normal form, we obtain

$$\Delta = {}^{16}/_9 s^2 f_e - {}^5/_3 \beta_i \beta_e - \beta_e^2 \tag{2.5}$$

$$\Delta \frac{d\alpha}{ds} = \frac{4}{3} s f_e \left(\frac{\alpha}{3} - \frac{\varphi}{f_e^{5/2}}\right) + \frac{2}{3}\beta_e \left(f_i - 9.48 \frac{\beta_e - \beta_i}{f_e^{3/2}}\right)$$

$$\Delta \frac{d\beta_i}{ds} = \frac{5}{3}\beta_e \beta_i \left(\frac{\alpha}{3} - \frac{\varphi}{f_e^{5/2}}\right) + \frac{\beta_e}{2sf_e}\left(\frac{16}{9} s^2 f_e - \beta_e^2\right)\left(f_i - 9.48 \frac{\beta_e - \beta_i}{f_e^{3/2}}\right)$$

$$\frac{d\beta_e}{ds} = \frac{4}{3} s \frac{d\alpha}{ds} - \frac{\alpha}{3} - \frac{d\beta_i}{ds}, \qquad \frac{df_e}{ds} = -\frac{\varphi}{\beta_e f_e^{3/2}}$$

$$\frac{d\varphi}{ds} = 2s \frac{df_e}{ds} - \beta_e \frac{d\alpha}{ds} - f_e - 9.48 \frac{\beta_e - \beta_i}{f_e^{3/2}}, \quad f_i = f_e \frac{\beta_i}{\beta_e}, \quad \delta = \frac{\beta_i}{f_i}$$

while it follows from conditions (2.2) and (2.3) that

$$\begin{gathered} \delta = \delta_N, \quad \alpha = \beta_e = \beta_i = f_e = f_i = \varphi = 0 \quad \text{when} \quad s = s_N \\ \alpha = \alpha_0, \quad \varphi = \varphi_0 \quad \text{when} \quad s = 0 \end{gathered} \tag{2.6}$$

where $s = s_N$ and $s = 0$ characterize, respectively, the position of the leading front of the disturbance and the position of the piston, and

$$\delta_N = \rho_N R^2 \varkappa_0^{-1} T_0^{-3/2}, \quad \alpha_0 = {}^4/_3 A R^{-1/2} T_0^{-1/2}, \quad \varphi_0 = 2BR^{1/2}\varkappa_0^{-1}T_0^{-3}$$

The following expression, connecting the linear coordinate with the similarity variables, may be obtained from the equation of continuity:

$$r(x, t) = \left(\frac{s}{\delta} + \frac{3}{4}\alpha\right) R^{1/2} T_0^{1/2} t^{4/3} \tag{2.7}$$

This implies, in particular, that the movement of the leading front of the disturbance is described by

$$r_N(t) = \frac{s_N}{\delta_N} R^{1/2} T_0^{1/2} t^{4/3} \tag{2.8}$$

3. It is clear from the boundary conditions (2.6) that $s = s_N$ is a singularity of system (2.5). Let us require that the transformation

$$\tau = (s_N - s)^{1/h}$$

where h is an integer greater than unity, is such that the required functions can be written in the neighborhood of $s = s_N$ as series in integral powers of τ. Retaining only the principal terms in the expansions, the asymptotic behavior of the functions are found to be given by

$$\begin{gathered} \alpha = \frac{3}{4}\frac{\delta_N}{s_N} f_e + \ldots, \qquad \delta = \delta_N + \frac{9}{16}\frac{\delta_N^3}{s_N^2} f_e + \ldots \\ \beta_e = \delta_N f_e + \ldots, \qquad \beta_i = 1.185 \frac{\delta_N^3}{s_N^2} f_e^2 + \ldots \\ f_i = 1.185 \frac{\delta_N^2}{s_N^2} f_e^2 + \ldots, \qquad \varphi = 2 s_N f_e + \ldots \\ f_e = \left[5 \frac{s_N}{\delta_N}(s_N - s)\right]^{2/5} + \ldots \end{gathered} \tag{3.1}$$

Expressions (3.1) for α, β_e, f_e, and φ, are the same as the asymptotic expressions for α, β, f, and φ in the neighborhood of the leading front of a disturbance obtained in [4], when solving the self-similar problem on a plane piston in the single-component medium, provided that n = 4/3 and that the thermal conductivity is proportional to the temperature to the power 5/2.

With small values of $s_N - s$, the determinant Δ of system (2.5) is positive, while at the piston, with $s = 0$, we have $\Delta < 0$. This implies (the proof is just the same as in [4]) that the solution of system (2.5) under boundary conditions (2.6) cannot be continued continuously from $s = s_N$ to $s = 0$. Here, we exclude the case in which at some point s_t at which $\Delta = 0$ the right-hand side of the first equation of system (2.5), and hence also, the right-hand side of the second equation of (2.5), vanishes.

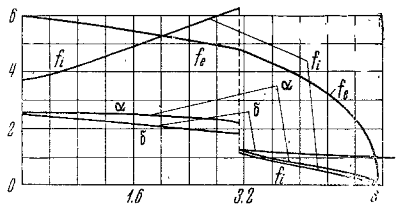

Fig. 1

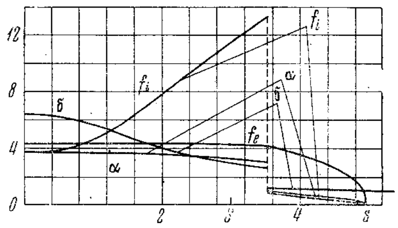

Fig. 2

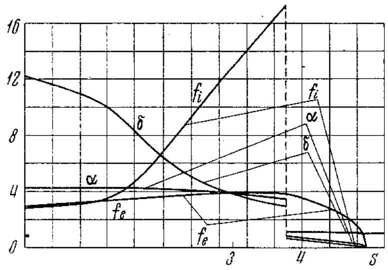

Fig. 3

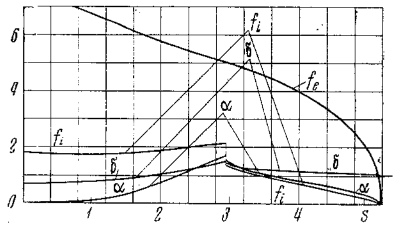

Fig. 4

With regard to the point s = 0, we merely observe that, when $f_e(0) > 0$, the singularity at this point can be eliminated.

4. Let the position of the surface of discontinuity of the hydrodynamic variables be characterized by the coordinate x_0, which corresponds to the value s_0 of the dimensionless variable s. The conditions on the isoelectron-thermal jump may be written as [3]

$$\rho_1\left(\frac{dr_0}{dt}-u_1\right)=\rho_2\left(\frac{dr_0}{dt}-u_2\right)=j, \qquad \theta_{e1}=\theta_{e2}$$

$$p_{i1}+p_{e1}+\rho_1\left(\frac{dr_0}{dt}-u_1\right)^2=p_{i2}+p_{e2}+\rho_2\left(\frac{dr_0}{dt}-u_2\right)^2 \tag{4.1}$$

$$\frac{1}{2}\left(\frac{dr_0}{dt}-u_1\right)^2+\frac{5}{2}\frac{p_{i1}}{\rho_1}-\frac{1}{j}W_1=\frac{1}{2}\left(\frac{dr_0}{dt}-u_2\right)^2+\frac{5}{2}\frac{p_{i2}}{\rho_2}-\frac{1}{j}W_2$$

$$W_1-W_2=j\frac{p_{e_1}}{\rho_1}\ln\frac{\rho_2}{\rho_1}$$

Here and below, subscripts 1 and 2 are assigned to function values on the "leading" and "trailing" front of the surface of discontinuity; dr_0/dt is the jump propagation speed, and

$$r_0(t)=\left(\frac{s_0}{\delta_1}+\frac{3}{4}\alpha_1\right)R^{1/3}T_0^{1/3}t^{4/3} \tag{4.2}$$

The laws of conservation of the mass and momentum flows at the jump have the usual form. In the law of conservation of the energy flow, we take account of the fact that $\theta_{e1}=\theta_{e2}$, and that the compression of the electron gas is isothermal.

Applying transformation (2.4) to (4.1), we can now express the quantities behind the jump in terms of

$$\varepsilon=\rho_1/\rho_2$$

As a result, we get

$$\alpha_2=\alpha_1+\frac{4}{3}(1-\varepsilon)\frac{s_0}{\delta_1}$$

$$f_{e2}=f_{e1}, \quad \beta_{e2}=\varepsilon^{-1}\beta_{e1} \tag{4.3}$$

$$\beta_{i2}=\beta_{i1}+\frac{1-\varepsilon}{\varepsilon}\left(\frac{16}{9}\varepsilon\frac{s_0^2}{\delta_1}-\beta_{e1}\right)$$

$$\varphi_2=\varphi_1-\frac{4}{3}s_0f_{e1}\ln\frac{1}{\varepsilon}, \quad f_{i2}=\varepsilon\frac{\beta_{i2}}{\delta_1}$$

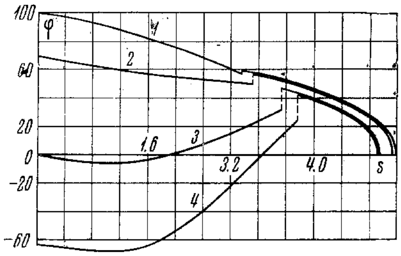

Fig. 5

The value of ε is found by solving the transcendental equation

$$(1-\varepsilon)\left[\frac{1+5(a_e+a_i)}{4}-\varepsilon\right]=\frac{a_e}{2}\ln\frac{1}{\varepsilon} \qquad (4.4)$$

$$a_e=\frac{9}{16}\frac{\delta_1}{s_0{}^2}\beta_{e1}, \qquad a_i=\frac{9}{16}\frac{\delta_1}{s_0{}^2}\beta_{i1}$$

The condition $\beta_{i_2} > \beta_{i_1}$ implies that the required root of (4.4) must satisfy the inequalities $a_e < \varepsilon < 1$.

It may easily be shown that only one root of (4.4) lies between a_e and 1.

5. Numerical integration of system (2.5) under boundary conditions (2.6) and conditions (4.3) on the jump was carried out by Luk'yanov.

The values of s_N and s_0 were fixed in some way and the Cauchy problem solved with the initial data, at a point where the asymptotic representation (3.1) still holds. The functions s_N and s_0 are the sums of the squares of the deviations of the given boundary conditions at the point s = 0 from the values obtained by solving the Cauchy problem. By using the method of fastest descent, we then found the values of s_N and s_0 at which the sum of squares of the deviations vanished.

The following typical cases were considered: a) a piston with a heat supply [$\varphi_0 = 69.91$, $\alpha_0 = 2.572$ (Fig. 1)], b) an adiabatic piston [$\varphi_0 = 0$, $\alpha_0 = 3.770$ (Fig. 2)], c) a piston with a heat drain [$\varphi_0 = -63.55$, $\alpha_0 = 4.308$ (Fig. 3)], and d) the problem of energy release at a fixed wall [$\varphi_0 = 100.9$, $\alpha_0 = 0$ (Fig. 4)].

The corresponding $\varphi(s)$ distributions are shown in Fig. 5; curves 1–4 correspond, respectively, to $\varphi_0 = 100.9$, 69.91, 0.0, −63.55; it was assumed throughout that $\delta_N = 1$.

It can be seen from the curves that the behavior of the hydrodynamic quantities is entirely typical of the motions considered in the electron-ion medium. A few points may be noticed. In the piston problem the ion temperature is greater than the electron temperature in the region immediately behind the jump, whereas the reverse is true in the problem of energy release at a fixed wall. The greater the energy supplied to the electron gas at the piston, the smaller the ion temperature jump. Finally, in all the cases considered, only flows corresponding to a temperature wave on the first kind [4] were obtained.

LITERATURE CITED

1. L. I. Sedov, Dimensional and Similarity Methods in Mechanics [in Russian], Gostekhizdat, Moscow, 1954.
2. S. I. Braginskii, "Transport phenomena in plasmas," collection: Aspects of Plasma Theory [in Russian], Gosatomizdat, Moscow, no. 1, 1963.
3. Ya. B. Zel'dovich and Yu. P. Raizer, Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena [in Russian], Fizmatgiz, Moscow, 1963.
4. P. P. Volosevich, S. P. Kurdyumov, L. N. Busurina, and V. P. Krus, "Solution of the one-dimensional plane problem on piston motion in an ideal heat-conducting gas," Zh. vychislit. matem. i matem. fiz., vol. 3, no. 1, 1963.